
GLOBAL EXISTENCE FOR ENERGY CRITICAL WAVES IN 3-D DOMAINS : NEUMANN BOUNDARY CONDITIONS

by

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Abstract. — We prove that the defocusing quintic wave equation, with Neumann boundary conditions, is globally wellposed on $H_N^1(\Omega) \times L^2(\Omega)$ for any smooth (compact) domain $\Omega \subset \mathbb{R}^3$. The proof relies on one hand on L^p estimates for the spectral projector ([13]), and on the other hand on a precise analysis of the boundary value problem, which turns out to be much more delicate than in the case of Dirichlet boundary conditions (see [3]).

1. Introduction

Let $\Omega \in \mathbb{R}^3$ be a smooth bounded domain with boundary $\partial\Omega$ and Δ_N the Laplacian acting on functions with Neumann boundary conditions. In [3] (with G. Lebeau), we derived Strichartz inequalities for the wave equation from L^p estimates for the associated spectral projector, obtained recently by H. Smith and C. Sogge [13]. As an application we obtained global well-posedness for the defocusing energy critical semi-linear wave equation (with real initial data) in Ω , with Dirichlet boundary conditions. Here we are interested in the similar question for the Neumann case,

$$(1.1) \quad \begin{aligned} (\partial_t^2 - \Delta)u + u^5 &= 0, & \text{in } \mathbb{R}_t \times \Omega \\ u|_{t=0} &= u_0, & \partial_t u|_{t=0} = u_1, & \partial_n u|_{\mathbb{R}_t \times \partial\Omega} = 0. \end{aligned}$$

Here and thereafter, ∂_n denotes the normal derivative to the boundary $\partial\Omega$. Equation (1.1) enjoys the usual conservation of energy

$$E(u)(t) = \int_{\Omega} \left(\frac{|\nabla u|^2(t, x) + |\partial_t u|^2(t, x)}{2} + \frac{|u|^6(t, x)}{6} \right) dx = E(u)(0) = E_0.$$

Let H_N^1 denote the Sobolev space associated to the Neumann Laplacian. Our main result reads:

Theorem 1. — *For any $(u_0, u_1) \in H_N^1(\Omega) \times L^2(\Omega)$ there exists a unique (global in time) solution u to (1.1) in the space*

$$X = C^0(\mathbb{R}_t; H_N^1(\Omega)) \cap C^1(\mathbb{R}_t; L^2(\Omega)) \cap L_{loc}^5(\mathbb{R}; L^{10}(\Omega)).$$

Remark 1. — Smith and Sogge's spectral projectors results hold irrespective of Dirichlet or Neumann boundary conditions. As such, the local existence result will be obtained as in [3]. However, the non-concentration argument requires considerably more care, including non-trivial L^p estimates for the traces of the solutions on the boundary. We provide a self-contained proof of the specific estimates we need, which applies to our nonlinear equation. A different set of results in that direction was announced in [17] in a more general setting for linear wave equations.

Remark 2. — Using the material in this paper, it is rather standard to prove existence of global smooth solutions, for smooth initial data satisfying compatibility conditions (see [12]). Furthermore, our arguments apply equally well to more general defocusing nonlinearities $f(u) = V'(u)$ satisfying

$$|f(u)| \leq C(1 + |u|^5), \quad |f'(u)| \leq C(1 + |u|)^4.$$

Finally, let us remark that our results can be localized (in space) and consequently hold also for a non compact domain and in the exterior of any obstacle, and we extend in this framework previous results obtained by Smith and Sogge [12] for convex obstacles (and Dirichlet boundary conditions).

For $s \geq 0$, let us denote by $H_N^s(\Omega)$ the domain of $(-\Delta_N)^{s/2}$. Finally, , we set $A \lesssim B$ to mean $A \leq CB$, where C is an harmless absolute constant (which may change from line to line).

2. Local existence

The local (in time) existence result for (1.1) is a direct consequence of some recent work by Smith and Sogge [13] on the spectral projector defined by $\Pi_\lambda = 1_{\sqrt{-\Delta_N} \in [\lambda, \lambda+1]}$.

Theorem A (Smith-Sogge [13, Theorem 7.1]). — *Let $\Omega \in \mathbb{R}^3$ be a smooth bounded domain and Δ be the Laplace operator with Dirichlet or Neumann boundary conditions, then there exists $C > 0$ such that for any $\lambda \geq 0$,*

$$(2.1) \quad \|1_{\sqrt{-\Delta_N} \in [\lambda, \lambda+1]} u\|_{L^5(\Omega)} \lesssim \lambda^{\frac{2}{5}} \|u\|_{L^2(\Omega)}.$$

From which we derive Strichartz estimates, which are optimal w.r.t scaling.

Theorem 2. — *Assume that for some $2 \leq q < +\infty$, the spectral projector Π_λ satisfies*

$$(2.2) \quad \|\Pi_\lambda u\|_{L^q(\Omega)} \lesssim \lambda^\delta \|u\|_{L^2(\Omega)}.$$

Then the solution to the wave equation $v(t, x) = e^{it\sqrt{-\Delta_N}} u_0$ satisfies

$$\|v\|_{L^q((0, 2\pi)_t \times \Omega_x)} \lesssim \|u_0\|_{H_N^{\delta + \frac{1}{2} - \frac{1}{q}}}.$$

The proof given in [3] applies verbatim and we therefore skip it.

Proposition 2.1. — *If u, f_1, f_2 satisfy*

$$(\partial_t^2 - \Delta)u = f_1 + f_2, \quad \partial_n u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1$$

then

$$(2.3) \quad \|u\|_{L^5((0,1);W_N^{\frac{3}{10},5}(\Omega))} + \|u\|_{C^0((0,1);H_N^1(\Omega))} + \|\partial_t u\|_{C^0((0,1);L^2(\Omega))} \\ \leq C \left(\|u_0\|_{H_N^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f_1\|_{L^1((0,1);L^2(\Omega))} + \|f_2\|_{L^{\frac{5}{4}}((0,1);W_N^{\frac{7}{10},\frac{5}{4}}(\Omega))} \right).$$

Furthermore, (2.3) holds (with the same constant C) if one replaces the time interval $(0, 1)$ by any interval of length smaller than 1.

Remark 3. — Notice that by Sobolev embedding and trace lemma, one has

$$(2.4) \quad \|u\|_{L^5((0,1);L^{10}(\Omega))} + \|u|_{\partial\Omega}\|_{L^5((0,1);L^{\frac{20}{3}}(\partial\Omega))} \lesssim \|u\|_{L^5((0,1);W_N^{\frac{3}{10},5}(\Omega))},$$

which will play a crucial role in the non-concentration argument.

Corollary 2.2. — For any initial data $(u_0, u_1) \in H_N^1(\Omega) \times L^2(\Omega)$, the critical non linear wave equation (1.1) is locally well posed in

$$X_T = C^0([0, T]; H_N^1(\Omega)) \cap L_{\text{loc}}^5((0, T); L^{10}(\Omega)) \times C^0([0, T]; L^2(\Omega))$$

(globally for small norm initial data).

We refer to Appendix A for a proof of Proposition 2.1. The proof of Corollary 2.2 proceeds by a standard fixed point argument with $(u, \partial_t u)$ in the space X_T with a sufficiently small T (depending on the initial data (u_0, u_1)). Note that this local in time result holds irrespective of the sign of the nonlinearity.

Finally, to obtain the global well posedness result for small initial data, it is enough to remark that if the norm of the initial data is small enough, then the fixed point can be performed in $X_{T=1}$. Then the control of the H^1 norm by the energy (which is conserved along the evolution) allows to iterate this argument indefinitely leading to global existence. Note that this result holds also irrespective of the sign of the nonlinearity because for small H^1 norms, the energy always control the H^1 norm.

3. Global existence

It turns out that our Strichartz estimates are strong enough to extend local to *global* existence for arbitrary (finite energy) data, when combined with trace estimates and non concentration arguments.

Before going into details, let us sketch the proof.

Remark that if $f = u^5$, we can estimate

$$(3.1) \quad \|u^5\|_{L^{\frac{5}{4}}((0,1);L^{\frac{30}{17}}(\Omega))} \leq \|u\|_{L^5((0,1);L^{10}(\Omega))}^4 \|u\|_{L^\infty((0,1);L^6(\Omega))}, \\ \|\nabla_x(u^5)\|_{L^{\frac{5}{4}}((0,1);L^{\frac{10}{9}}(\Omega))} = 5\|u^4 \nabla_x u\|_{L^{\frac{5}{4}}((0,1);L^{\frac{10}{9}}(\Omega))} \\ \leq 5\|u\|_{L^5((0,1);L^{10}(\Omega))}^4 \|u\|_{L^\infty((0,1);H^1(\Omega))}.$$

Interpolating between these two inequalities yields

$$(3.2) \quad \|u^5\|_{L^{\frac{5}{4}}((0,1);W_N^{\frac{7}{10},\frac{5}{4}}(\Omega))} \lesssim \|u\|_{L^5((0,1);L^{10}(\Omega))}^4 \|u\|_{L^\infty((0,1);L^6(\Omega))}^{\frac{3}{10}} \|u\|_{L^\infty((0,1);H^1(\Omega))}^{\frac{7}{10}}.$$

Following ideas of Struwe [14], Grillakis [5] and Shatah-Struwe [10, 11], we will localize these estimates on small light cones and use the fact that the $L_t^\infty; L_x^6$ norm is small in such small cones. Unlike with Dirichlet boundary conditions, we cannot hope to have a good control of the H^1 norm of the trace of u on the boundary (this control is known to be false even for the linear problem), and it will require a more delicate argument to handle boundary terms.

3.1. The L^6 estimate. — In this section we shall always consider solutions to (1.1) in

$$(3.3) \quad X_{<t_0} = C^0([0, t_0]; H_N^1(\Omega)) \cap L_{loc}^5([0, t_0]; L^{10}(\Omega)) \times C^0([0, t_0]; L^2(\Omega)).$$

Moreover, we assume these solutions to have energy bounded by a fixed constant (namely, E_0). By a standard procedure, such solutions are obtained as limits in $X_{<t_0}$ of smooth solutions to the analog of (1.1) where the nonlinearity and the initial data have been smoothed out. Consequently all the subsequent integrations by parts will be licit by a limiting argument.

3.1.1. The flux identity. — By time translation, we shall assume later that $t_0 = 0$. Let us first define

$$(3.4) \quad \begin{aligned} Q &= \frac{|\partial_t u|^2 + |\nabla_x u|^2}{2} + \frac{|u|^6}{6} + \partial_t u \left(\frac{x}{t} \cdot \nabla_x \right) u, \\ P &= \frac{x}{t} \left(\frac{|\partial_t u|^2 - |\nabla_x u|^2}{2} - \frac{|u|^6}{6} \right) + \nabla_x u \left(\partial_t u + \left(\frac{x}{t} \cdot \nabla_x \right) u + \frac{u}{t} \right), \end{aligned}$$

$$\begin{aligned} \Omega_S^T &= [S, T] \times \Omega, & \partial\Omega_S^T &= [S, T] \times \partial\Omega \\ D_T &= \{x; |x| < -T\}, & M_S^T &= \{x; |x| = -t\} \cap \Omega_S^T \\ K_S^T &= \{(x, t); |x| < -t\} \cap \Omega_S^T \\ \partial K_S^T &= (\partial\Omega_S^T \cap K_S^T) \cup D_T \cup D_S \cup M_S^T \\ e(u) &= \left(\frac{|\partial_t u|^2 + |\nabla_x u|^2}{2} + \frac{|u|^6}{6}, -\partial_t u \nabla_x u \right) \end{aligned}$$

and the flux across M_S^T

$$\text{Flux } (u, M_S^T) = \int_{M_S^T} \langle e(u), \nu \rangle d\rho(x, t),$$

where

$$(3.5) \quad \nu = \frac{1}{\sqrt{2}|x|}(-t, -\frac{x}{t}) = \frac{1}{\sqrt{2}|x|}(|x|, \frac{x}{|x|})$$

is the outward normal to M_S^T and $d\rho(x, t)$ the induced measure on M_S^T . Remark that

$$(3.6) \quad \begin{aligned} \text{Flux } (u, M_S^T) &= \int_{M_S^T} \frac{|\partial_t u|^2 + |\nabla_x u|^2}{2} + \frac{|u|^6}{6} - \partial_t u \frac{x}{|x|} \cdot \nabla_x u d\rho(x, t) \\ &= \int_{M_S^T} \frac{1}{2} \left| \frac{x}{|x|} \partial_t u - \nabla_x u \right|^2 + \frac{|u|^6}{6} d\rho(x, t) \geq 0. \end{aligned}$$

One may notice that the boundary of K_S^T is $M_S^T \cup \partial\Omega_S^T \cup D_S \cup D_T$, and that, due to the boundary condition, there is no flux through $\partial\Omega_S^T$.

An integration by parts gives (see Rauch [9] or [12, (3.3')])

$$(3.7) \quad \int_{x \in \Omega, |x| < -T} \left(\frac{|\partial_t u|^2 + |\nabla_x u|^2}{2} + \frac{|u|^6}{6} \right) (x, T) dx + \text{Flux}(u, M_S^T) \\ = \int_{x \in \Omega, |x| < -S} \left(\frac{|\partial_t u|^2 + |\nabla_x u|^2}{2} + \frac{|u|^6}{6} \right) (x, S) dx = E_{loc}(S).$$

Therefore $u|_{M_S^T}$ is bounded in $H^1(M_S^T) \cap L^6(M_S^T)$ (uniformly with respect to $T < 0$), and $E_{loc}(S)$ being a non-negative non-increasing function, it has a limit when $S \rightarrow 0^-$, and

$$(3.8) \quad \text{Flux}(u, M_S^0) = \lim_{T \rightarrow 0^-} \text{Flux}(u, M_S^T) = \lim_{T \rightarrow 0^-} (E_{loc}(S) - E_{loc}(T))$$

exists and satisfies

$$(3.9) \quad \lim_{S \rightarrow 0^-} \text{Flux}(u, M_S^0) = 0.$$

3.1.2. A priori estimate for traces. — We prove an a priori estimate on finite energy solutions of (1.1), which is reminiscent of results obtained in [17] for variable coefficients linear wave equations. Observe that, unlike in the Dirichlet case, we don't have any uniform Lopatinskii condition, which prevents control of the gradient on the boundary. The following result will provide a substitute. It shows that even though (due to the failure of the uniform Lopatinskii condition), the H^1 norm of the trace is known to be unbounded (at least for the linear equation), the integral on the boundary of a specific quadratic form (namely the so-called Q_0 null form) remain bounded (see [17] for related results).

Proposition 3.1. — *Let $n(x)$ be the exterior unit normal vector to $\partial\Omega$, $-\varepsilon < S < T \leq 0$, $d\sigma$ the induced measure on $\partial\Omega$, and u a solution to (1.1). Then*

$$(3.10) \quad \left| \int_{K_S^T \cap \partial\Omega_S^T} (|\partial_t u|^2 - |\nabla u|^2 - \frac{u^6}{3}) n(x) \cdot x d\sigma dt \right| \lesssim |S|^2 (E_0 + E_0^{\frac{2}{3}}),$$

where the tip of the cone K_S^0 is located on $\partial\Omega$ and has been set to be $0 \in \partial\Omega$.

Remark 4. — For the solution of the linear wave equation, $\square u = 0$, taking the trace on the boundary $\partial\Omega$ in (2.3) gives

$$(3.11) \quad \|u|_{\partial\Omega}\|_{L^5((0,1); W^{\frac{1}{10},5}(\partial\Omega))} + \|u|_{\partial\Omega}\|_{L^\infty((0,1); H^{1/2}(\partial\Omega))} \\ \leq C \|u\|_{L^5((0,1); W_N^{\frac{3}{10},5}(\Omega))} + \|u\|_{L^\infty((0,1); H^1(\Omega))} \leq C \left(\|u_0\|_{H_N^1(\Omega)} + \|u_1\|_{L^2(\Omega)} \right)$$

And interpolating between the two estimates in the l.h.s. of (3.11) gives

$$\|u|_{\partial\Omega}\|_{L^6((0,1) \times \partial\Omega)} \leq C \left(\|u_0\|_{H_N^1(\Omega)} + \|u_1\|_{L^2(\Omega)} \right)$$

As a consequence, one could infer that the u^6 term in the r.h.s. of (3.10) is not the most important part of the estimate. However, we will not use this fact here and shall prove the estimate (3.10) as a whole

Proof. — The main step in the proof of Proposition 3.1 is an integration by parts using suitable weight and vector field.

Lemma 3.2. — *There exists a smooth vector field Z (depending on (t, x)) defined on a neighborhood of x_0 in $[-t_0, 0) \times \overline{\Omega}$ such that*

1. *We have $Z|_{\partial\Omega} = \partial_n + \tau$ where ∂_n is the interior normal derivative to the boundary and τ is a vector field which is tangent to the boundary;*
2. *The restriction of Z to the sphere $S_t = \{x \in \Omega; |x| = -t\}$ is tangent to that sphere.*

Proof. — Performing a linear orthogonal change of variables, we can assume that $\partial_3 = \partial_{n(0)}$, with $n(0)$ the normal vector to $\partial\Omega$ at $x = 0$. We now build a vector field $Z \in C^\infty(\Omega; T\Omega)$ (as such, it does not have a ∂_t component) such that its restriction to $\partial\Omega$ is close to ∂_n (which, in a neighborhood of $x = 0$, $B(0, \eta)$, is essentially ∂_3 up to an $O(\eta)$ term).

Consider the half-sphere $x_3 = \sqrt{1 - r^2}$, with $r^2 = x_1^2 + x_2^2$. To define a vector field X in the zone $x_3 < \frac{3}{4}$ (thus multiplying by a cut-off $\phi(x)$ such that $\phi = 0$ for $x > 3/4$ and $\phi = 1$ for $x < 1/4$), we set $X = X_1 = \partial_3$ for $r < 1/4$; for $r > 3/4$, we set $X = X_2 = \partial_3 - r^{-1}x_3\partial_r$. On the half-sphere, X will be tangent and pointing in the same direction as ∂_3 (or ∂_ν). Now, we may smoothly connect both zones and define X within the entire half-sphere: consider $\phi \in C_0^\infty(-1/2, 1/2)$ equal to 1 on $(0, 1/4)$ and

$$X = \phi(x_3)(\phi(r)\partial_3 + (1 - \phi(r))(\partial_3 - \frac{x_3}{r}\partial_r)).$$

Then one rescales (x_3, r) to be $(x_3/t, r/t)$. Note that on the $x_3 = 0$ axis, $X = \partial_3$. We now set

$$X_t = \phi(x_3/t)(\phi(r/t)X_1(x/t) + (1 - \phi(r/t))X_2)(x/t).$$

Remark that X_1, X_2 , as previously defined, are time independent (and smooth in zones where $X \neq 0$). The vector field X_t is tangent to the sphere S_t by definition. However, it requires to be renormalized so that its component on the normal to the spatial boundary is one: define $z(x) \in \partial\Omega$ to be the orthogonal projection to the (spatial) boundary, we set

$$Z = \frac{1}{X_t(z(x)) \cdot n(z(x))} X_t,$$

which is easily seen to satisfy all our requirements. □

Now we define a smooth weight $w(x)$, such that its restriction to the boundary $\partial\Omega$ is $x \cdot n(x)$, and such that $|w(x)| = O(|r|^2)$: $w(x) = z(x) \cdot n(z(x))$ is such a weight.

To prove Proposition 3.1, we start with the following commutator estimate.

Proposition 3.3. — *There exists $C > 0$ such that*

$$(3.12) \quad \left| \int_{K_S^T \cap \Omega_S^T} [(\partial_t^2 - \Delta), Z]u(t, x) \cdot u(t, x)w(x)dxdt \right| \lesssim |S|^2(E_0 + E_0^{\frac{2}{3}}).$$

Proof. — We shall make a repeated use of the following

Lemma 3.4. — *Consider a function $a(x, t)$ such that*

$$\forall(x, t); |x| \leq -t, t \in (-1, 0), \quad |a(x, t)| \leq M.$$

Then there exists $C > 0$ such that for any solution with energy E_0 , for all S, T s.t. $-1 \leq S < T < 0$,

$$(3.13) \quad \left| \int_{K_S^T} a(x, t) u(x, t) \nabla_{x,t} u(x, t) dx dt \right| \lesssim E_0^{\frac{2}{3}} |S|^2.$$

Proof. — We have

$$(3.14) \quad \begin{aligned} & \left| \int_{K_S^T} a(x, t) u(x, t) \nabla_{x,t} u(x, t) dx dt \right| \\ & \leq \left(\int_S^T |\nabla_{x,t} u(x, t)|^2 dx dt \right)^{1/2} \left(\int_{K_S^T} |u(x, t)|^6 dx dt \right)^{1/6} \left(\int_{K_S^T} |a(x, t)|^3 dx dt \right)^{1/3} \\ & \lesssim E^{\frac{1}{2} + \frac{1}{6}}(u) |S|^{\frac{1}{2} + \frac{1}{6}} \left(\int_{K_S^T} |a(x, t)|^3 dx dt \right)^{1/3} \\ & \lesssim (E_0 |S|)^{2/3} \left(\int_S^T \int_{|x| < -t} dx dt \right)^{1/3} = C E_0^{\frac{2}{3}} |S|^2, \end{aligned}$$

which achieves the proof. \square

We now prove a second lemma which will immediately yield Proposition 3.3:

Lemma 3.5. — Assume that $Q = \sum_{ij} q_{ij} \partial_{ij}^2$ is a smooth (space-time) second order operator with real coefficients $(q_{ij})_{ij}$, $0 \leq i, j \leq 3$, satisfying

$$(3.15) \quad \forall i, j, \forall (x, t); |x| \leq -t, t \in (-1, 0), \quad |\partial_{x,t}^\alpha q_{ij}(x, t)| \lesssim |t|^{1-|\alpha|}, \quad |\alpha| \leq 2.$$

Moreover, assume that (recalling (3.5) for the definition of ν)

$$(3.16) \quad T = \sum_{i,j} ((\nu \cdot \partial_i) q_{ij}) \partial_j \text{ is a vector field which is tangent to } M_S^T;$$

then, for any solution with energy E_0 , and any $-1 \leq S < T < 0$,

$$(3.17) \quad \left| \int_{K_S^T} Q(x, t, D_{x,t}) u(x, t) u(x, t) dx dt \right| \lesssim |S|^2 (E_0 + E_0^{\frac{2}{3}}).$$

Proof. — We integrate by parts

$$\begin{aligned}
(3.18) \quad & \int_{K_S^T} Q(x, t, D_{x,t}) u(x, t) u(x, t) dx dt \\
&= - \sum_{i,j} \int_{K_S^T} \partial_i(q_{i,j}(x, t)) \partial_j u(x, t) u(x, t) dx dt \\
&\quad - \sum_{i,j} \int_{K_S^T} q_{i,j}(x, t) \partial_j u(x, t) \partial_i(u(x, t)) dx dt \\
&\quad + \sum_{i,j} \left[\int_{D_T} q_{0,j}(x, t) \partial_j u(x, t) u(x, t) dx - \int_{D_S} q_{0,j}(x, t) \partial_j u(x, t) u(x, t) dx \right] \\
&\quad + \sum_{i,j} \int_{M_S^T} (\nu(x, t) \cdot \partial_i) q_{i,j}(x, t) \partial_j u(x, t) u(x, t) d\rho \\
&\quad + \sum_{i,j} \int_{K_S^T \cap \partial\Omega_S^T} (-n(x) \cdot \partial_i) q_{i,j}(x, t) \partial_j u(x, t) u(x, t) d\sigma dt \\
&= I + II + III + IV + V,
\end{aligned}$$

where we recall $\nu(x, t) = \frac{1}{\sqrt{2}}(\partial_r + \partial_t)$ to be the outward normal vector to M_S^T and $n(x)$ to be the inward normal vector to $\partial\Omega$, while $d\rho$ (resp. $d\sigma$) is the induced measure on M_S^T (resp. $\partial\Omega$).

The contribution of I is dealt with using Lemma 3.4. Next,

$$II \lesssim \|q_{i,j}\|_{L^\infty(K_S^T)} |S| E(u) \lesssim |S|^2 E_0.$$

The contribution of D_S in III is bounded (using Hölder inequality) by

$$\begin{aligned}
(3.19) \quad & \left(\int_{D_S} |\nabla_{x,t} u(x, t)|^2 dx dt \right)^{1/2} \left(\int_{D_S} |u(x, t)|^6 dx dt \right)^{1/6} \left(\int_{D_S} |q_{i,j}(x, t)|^3 dx \right)^{1/3} \\
& \lesssim E(u)^{2/3} \left(\int_{D_S} |q_{i,j}(x, S)|^3 dx \right)^{1/3} \lesssim E_0^{2/3} |S| \left(\int_{|x| < |s|} dx \right)^{1/3} = C E_0^{2/3} |S|^2.
\end{aligned}$$

We deal with the contribution of D_T to III similarly. To bound the contribution of IV , we remark that according to our assumptions on Q , the vector field

$$\frac{1}{\sqrt{2}} T = \sum_{i,j} (\nu(x, t) \cdot \partial_i) q_{i,j}(x, t) \partial_j$$

is tangential to M_S^T . As a consequence (using Hölder inequality),

$$\begin{aligned}
(3.20) \quad IV & \lesssim \left(\int_{M_S^T} |\nabla_{tan} u(x, t)|^2 d\rho \right)^{1/2} \left(\int_{M_S^T} |u(x, t)|^6 d\rho \right)^{1/6} \left(\int_{M_S^T} |q_{i,j}(x, t)|^3 d\rho \right)^{1/3} \\
& \lesssim (\text{Flux}(u, M_S^T))^{2/3} \left(\int_{M_S^T} |q_{i,j}(x, t)|^3 d\rho \right)^{1/3} \\
& \lesssim (\text{Flux}(u, M_S^T))^{2/3} |S|^2 \lesssim E_0^{2/3} |S|^2
\end{aligned}$$

where in the last inequality we used (3.7).

It remains to bound the contributions of V in the right hand side of (3.18). Using the Neumann boundary condition, we can replace the vector fields ∂_j by their tangential components $\partial_j - (\partial_j \cdot n(x))n(x)$. In a coordinate system y in the boundary and abusing notation for ∂_{y_i} , we now have to compute

$$(3.21) \quad V = \int_{K_S^T \cap \partial\Omega} \tilde{q}_{i,j}(y, t) \partial_j u(y, t) u(y, t) dy dt$$

with \tilde{q} satisfying the same estimates as q , namely (3.15). Integrating by parts gives

$$(3.22) \quad V = \frac{1}{2} \int_{K_S^T \cap \partial\Omega_S^T} \partial_j(\tilde{q}_{i,j}(y, t)) |u(y, t)|^2 d\sigma dt + \frac{1}{2} \int_{M_S^T \cap \partial\Omega_S^T} \tilde{q}_{i,j}(y, t) |u(y, t)|^2 d\rho = V.1 + V.2,$$

where the first term in the right hand side appears when the derivative hits the coefficients while the second term comes from the contribution of the boundary of $K_S^T \cap \partial\Omega_S^T$.

To conclude, we use the following

Lemma 3.6. — *For any solution u and any $S < T < 0$, we have*

$$(3.23) \quad \|u|_{M_S^T \cap \partial\Omega_S^T}\|_{L^4(M_S^T \cap \partial\Omega_S^T)}^2 \lesssim \text{Flux}(u, M_S^T) \lesssim E_0.$$

Proof. — Let us first conclude the proof of Lemma 3.5 assuming Lemma 3.6. To deal with the first term in the right hand side of (3.22), we use Hölder inequality and obtain (using the trace theorem and then Sobolev embedding on $\partial\Omega$)

$$(3.24) \quad |V.1| \lesssim \left(\int_{\partial\Omega_S^T} |u|^4 d\sigma dt \right)^{\frac{1}{2}} \left(\int_{K_S^T \cap \partial\Omega_S^T} |\partial_j(\tilde{q}_{i,j}(x, t))|^2 \right)^{\frac{1}{2}} \\ \lesssim |S|^{\frac{3}{2} + \frac{1}{2}} \sup_t \left(\int_{\partial\Omega} |u|^4 d\sigma \right)^{\frac{1}{2}} \lesssim |S|^2 E_0.$$

To deal with the second term in the right hand side of (3.22), we also use Hölder inequality and obtain (using Lemma 3.6)

$$(3.25) \quad |V.2| \lesssim \left(\int_{M_S^T \cap \partial\Omega_S^T} |u(x, t)|^4 d\sigma \right)^{1/2} \left(\int_{M_S^T \cap \partial\Omega_S^T} |(\tilde{q}_{i,j}(x, t))|^2 \right)^{1/2} \lesssim |S|^2 E_0.$$

which ends the proof of Lemma 3.5 □

Let us now prove Lemma 3.6. The idea is roughly to take benefit of the fact that the flux controls the H^1 norm on the cone M_S^T ; consequently by trace theorems it controls the $H^{1/2}$ norm of the restriction of u on the intersection of the cone and the spatial boundary. Hence, by Sobolev's theorem, it controls the L^4 norm. However, the geometry of the cone becomes singular near $t = 0$ and we have to be careful when implementing this idea. For this we decompose the time interval (S, T) into a union of dyadic intervals

$$(S, T) = \cup_{j=j_0}^{j_1-1} (t_j, t_{j+1}), \quad t_j = -2^{-j}$$

(assuming for simplicity that $S = -2^{-j_0}$ and $T = -2^{-j_1}$), and we decompose the integral (3.23) into

$$\sum_{j=j_0}^{j_1-1} \|u|_{M_S^T \cap \partial\Omega_S^T}\|_{L^4(M_S^T \cap \partial\Omega_S^T)}^4$$

For each integral, we perform the change of variables

$$(t, x) \mapsto (s = 2^j t, y = 2^j x), u_j(s, y) = u(2^j s, 2^j y)$$

and we have to bound for any $j_0 \leq j \leq j_1 - 1$

$$\|(u_j)_{|2^j(M_{-2^{-j}}^{-2-j-1} \cap \partial\Omega_{-2^{-j}}^{-2-j-1})}\|_{L^4}^4$$

Observe now that

$$2^j(M_{-2^{-j-1}}^{-2-j-2} \cap \partial\Omega_{-2^{-j-1}}^{-2-j-2})$$

is a smooth family of hypersurfaces of the cone (included in the smooth part of the cone corresponding to $\{t \in (1, 1/2)\}$) and consequently, the trace theorem applies with uniform constants; tracking the change of variables yields

$$\|u_{|(M_{-2^{-j}}^{-2-j-1} \cap \partial\Omega_{-2^{-j}}^{-2-j-1})}\|_{L^4}^4 \lesssim \|u_{|(M_{-2^{-j-1}}^{-2-j-2} \cap \partial\Omega_{-2^{-j-1}}^{-2-j-2})}\|_{H^1}^4.$$

Summing the pieces back provides the desired estimate. \square

Back to proving Proposition 3.3, one may easily see that the coefficients of the second order operator $w(x)[\square, Z]$ satisfy the decay conditions of Lemma 3.4 and 3.5. We are left with (3.16): from decomposing the Laplacian in polar coordinates, the first order term $(r^{-1}\partial_r)$ is harmless, and so is the angular part. Then, if $A = \partial_t^2 - \partial_r^2 = (\partial_t + \partial_r)(\partial_t - \partial_r)$, $w(x)[A, Z]$ doesn't have a $(\partial_t + \partial_r)^2$ component, hence it satisfies (3.16). \square

Having proved that

$$(3.26) \quad \left| \int_{K_S^T \cap \Omega} [(\partial_t^2 - \Delta), Z] u(t, x) u(t, x) w(x) dx dt \right| \lesssim S^2 (E_0 + E_0^{\frac{2}{3}}),$$

we may return to the proof of Proposition 3.1 and perform integration by parts with the \square operator in (3.26). Denote by $D = \Omega_S^T \cap K_S^T$ the space-time domain of integration. Its boundary ∂D will be the reunion of two time slices, $D_T \cap \Omega$ and $D_S \cap \Omega$, one space slice (on the boundary) $\partial\Omega_S^T \cap K_S^T$ and the cone boundary inside the domain, $\Omega_S^T \cap M_S^T$.

As

$$(3.27) \quad \int_D [(\partial_t^2 - \Delta), Z] u(t, x) \cdot u(t, x) w(x) dx dt = \int_D Z(u^5) u(t, x) w(x) dx dt \\ + \int_D (\partial_t^2 - \Delta) Z(u) \cdot u(t, x) w(x) dx dt,$$

Leaving aside the first term, we compute the remaining term, namely

$$\int_D w(x, t) u \square Z u dx dt.$$

We will apply Green's second identity: recall that over 4-dimensional domains V , with $\nu_{\partial V}$ the outward normal to the boundary,

$$(3.28) \quad \int_V \phi \square \psi - \psi \square \phi = \int_{\partial V} \left(\psi \left(\frac{-\partial_t \phi}{\nabla \phi} \right) \cdot \nu_{\partial V} - \phi \left(\frac{-\partial_t \psi}{\nabla \psi} \right) \cdot \nu_{\partial V} \right).$$

Let us start with $Zu \square (uw) = Zu(u \square w + w \square u + Q_0(w, u))$: The last term is controlled by $O(S^2)E(u)$: in fact, one has 2 derivatives hitting the u 's (one from Z and one from Q_0), and the remaining one from Q_0 on w , which loses an S ; and then integration over t which

regains the lost factor S . The first term is no worse: one loses both factors S in deriving w twice, but on the other hand we get

$$\int_{D_t} u^2 dx \leq \left(\int_{D_t} dx \right)^{\frac{2}{3}} \left(\int_{D_t} u^6 dx \right)^{\frac{1}{3}} \lesssim S^2 E(u)^{\frac{1}{3}},$$

since the volume of the ball D_t is t^3 which produces the S^2 factor. The middle term is, after substitution,

$$\int_D w(x) Z u(x, t) (-u^5) dx dt,$$

which may be added to the term we left in (3.27), to get

$$\begin{aligned} (3.29) \quad 4 \int_D (Zu) u^5 w(x) dx dt &= \frac{2}{3} \int_D Z(u^6) w(x) dx dt \\ &= -\frac{2}{3} \int_D u^6 Z(w) dx dt - \frac{2}{3} \int_{\partial \Omega_S^T \cap K_S^T} u^6 w d\sigma, \end{aligned}$$

as Z is tangent to both the time slices and the cone (hence, the boundary contributions of these regions vanish). Recalling that $Z(w) = O(S)$,

$$\left| \int_{K_S^T \times \Omega} u^6 Z w(x) dx dt \right| \lesssim S^2 E(u),$$

and we collect a term

$$(3.30) \quad -\frac{2}{3} \int_{\partial \Omega_S^T \cap K_S^T} u^6 w d\sigma,$$

for later use.

We are now left with a space-time boundary term J coming from our application of (3.28), which is a difference of two terms

$$(3.31) \quad J = \int_{\partial D} (uw) N \cdot \left(\frac{\partial_t}{-\nabla} \right) Zu - Zu N \cdot \left(\frac{\partial_t}{-\nabla} \right) (uw) = J_1 - J_2.$$

Here N is the outward normal derivative to the boundary ∂D . The second term J_2 splits itself in three sub-terms.

- The first two are boundary terms on $M_S^T \cap \Omega$ and $D_S \cup D_T$. The cone term is controlled by S^2 times the flux, as one has only tangential derivatives on u (either Z or $L = \sqrt{2}^{-1}(\partial_t - \partial_r)$). The time-slice terms are similarly controlled by S^2 times the energy.
- The last one is

$$\int_{\partial \Omega_S^T \cap K_S^T} u Z u \partial_n w + w Z u \partial_n u d\sigma dt,$$

and both terms vanish: the first one because $\partial_n w = 0$ and the second one because of the Neumann boundary condition.

We are thus left with J_1 , which we split again in three terms

$$\begin{aligned}
J_1 &= \int_{M_S^T} wu(LZu) d\rho \\
&\quad + \int_{D_T \cap \Omega} wu \partial_t Zu dx - \int_{D_S \cap \Omega} wu \partial_t Zu dx \\
&\quad + \int_{K_S^T \cap \partial\Omega_S^T} (\partial_n Zu) uw d\sigma dt \\
&= K_1 + K_2 + K_3.
\end{aligned}$$

Consider K_3 : we chose Z such that $Z|_{\partial\Omega} = \partial_n + \tau$, where τ is a tangent vector field to $\partial\Omega$, so that

$$\begin{aligned}
K_3 &= \int_{K_S^T \cap \partial\Omega_S^T} uw \partial_n^2 u + wu[\partial_n, \tau]u d\sigma dt \\
&= \int_{K_S^T \cap \partial\Omega_S^T} uw (\partial_t^2 - \Delta_{\tan})u + u^6 w d\sigma dt + \int_{K_S^T \cap \partial\Omega_S^T} wu[\partial_n, \tau]u d\sigma dt.
\end{aligned}$$

We integrate by parts, to get

$$\begin{aligned}
\int_{K_S^T \cap \partial\Omega_S^T} uw (\partial_t^2 - \Delta_{\tan} + u^4)u d\sigma dt &= \int_{K_S^T \cap \partial\Omega_S^T} (|\nabla_{\tan} u|^2 - |\partial_t u|^2 + u^6)w d\sigma dt \\
&\quad + \int_{K_S^T \cap \partial\Omega_S^T} u \partial_{\tan} u (\eta w + \partial_{\tan} w) d\sigma dt + B,
\end{aligned}$$

where B is the boundary term to be specified below. The second term can be dealt with as in (3.20): η is the coefficient in $[\partial_n, \tau]$ which is tangent again (certainly, at least, when applied to u thanks to $\partial_n u = 0$!). Now,

$$(3.32) \quad B = \int_{\partial\Omega_S^T \cap M_S^T} wu N_2 \cdot \left(-\nabla_{\tan} \right) u d\sigma_2,$$

where N_2 is the normal to $\partial\Omega \cap K_S^T$ in $\partial\Omega$ and $d\sigma_2$ the induced (2D) measure; we leave B for later treatment.

Adding the first term and the term (3.30), we get the left-handside of (3.10), namely

$$\int_{K_S^T \cap \partial\Omega_S^T} (|\nabla_{\tan} u|^2 - |\partial_t u|^2 + \frac{u^6}{3})w d\sigma dt$$

We now return to K_2 , writing (with an obvious abuse of notation for the domains)

$$\begin{aligned}
K_2 &= \int_{D_T \setminus D_S} w(u[Z, \partial_t]u - \partial_t u Zu) dx \\
&\quad + \int_{(D_T \setminus D_S) \cap \partial\Omega} wu(N \cdot Z) \partial_t u d\sigma \\
&\quad + \int_{(D_T \setminus D_S) \cap M_S^T} wu(N \cdot Z) \partial_t u.
\end{aligned}$$

The last term vanishes, as Z is tangent to M_S^T . The first one is controlled by $S^2 E_0$. We collect the second one, denoted by M , for later treatment: recalling that on that part of ∂D , $N = -\partial_n$, we have $Z \cdot N = -Z \cdot \partial_n = -1$ and

$$M = - \int_{(D_T \setminus D_S) \cap \partial \Omega} wu \partial_t u \, d\sigma.$$

We now return to K_1 :

$$\begin{aligned} K_1 &= \int_{\Omega_S^T \cap M_S^T} wu [L, Z] u \, d\rho \\ &\quad + \int_{\Omega_S^T \cap M_S^T} wu Z L u \, d\rho. \end{aligned}$$

The commutator between two tangent vector fields is tangent, hence the first term is controlled through the flux like in (3.20). By integration by parts, the second term is

$$- \int_{\Omega \cap M_S^T} Z(wu) L u \, d\rho + \int_{\partial(\Omega \cap M_S^T)} wu L u (Z \cdot N_1) \, d\sigma_2,$$

where N_1 is the normal to $\Omega \cap M_S^T$ in M_S^T , and $d\sigma_2$ the measure on the boundary. The first term is again controlled through the flux, and we are left with the second one, namely

$$P = \int_{\partial(\Omega \cap M_S^T)} wu L u (Z \cdot N_1) \, d\sigma_2.$$

At this point, the only hope is that B, M and P will cancel each other, as they are integrals over a two-dimensional set which is the intersection of M_S^T and $\partial \Omega_S^T$. We decompose these into three distinct regions:

- on $D_T \cap \partial \Omega$, we get $-wu \partial_t u$ from M and $wu \partial_t u$ from B as on this part, $N_2 = \partial_t$. Hence, the total contribution vanishes. The same thing (with $N_2 = -\partial_t$) applies to $D_S \cap \partial \Omega$.
- On $(D_T \setminus D_S) \cap M_S^T$, one gets only a contribution from P : however, on this part of the boundary, $N_1 = L$, and Z is tangent to the cone with no time component, hence $N_1 \cdot Z = 0$ and this term vanishes as well.
- Finally, we are left with $\partial \Omega \cap M_S^T$. we get non zero contributions from B and P : both have a factor wu , the same measure, and the terms (recall for the first one that $\partial_n u = 0$!) are equal (respectively) to

$$N_2 \cdot \begin{pmatrix} \partial_t \\ -\nabla \end{pmatrix} u \text{ and } (Z \cdot N_1) \nu \cdot \begin{pmatrix} \partial_t \\ -\nabla \end{pmatrix} u,$$

where, once again, $\nu = \frac{\partial_t + \partial_r}{\sqrt{2}}$ is the normal vector to the cone.

On the (2-dimensional) edge over which the integration is performed, we denote by T the projection of Z : there are two different ways of writing Z on a direct orthonormal basis on the edge:

- we see the edge as the boundary on B and use

$$Z = T + \partial_n + (Z \cdot N_2) N_2;$$

- we see the edge as the boundary on P and use

$$Z = T + (Z \cdot N_1) N_1 + (Z \cdot \nu) \nu.$$

From our choice of Z , the very last term in the second decomposition vanishes. On the other hand, we have a two-dimensional hyperplane where $Z - T$ lives, with two different basis, $\{-\partial_n, N_2\}$ and $\{N_1, \nu\}$ (where the $-$ in front of ∂_n results from our choice of ∂_n as the inward normal direction to $\partial\Omega$, while N_1 is the outward normal to $M_S^T \cap \partial\Omega_S^T$). Therefore, $\nu \cdot N_2 = -N_1 \cdot \partial_n$. Now, we have

$$\nu = \lambda \partial_n + \mu N_2, \text{ and } \nu \cdot \begin{pmatrix} \partial_t \\ -\nabla \end{pmatrix} u = \mu N_2 \cdot \begin{pmatrix} \partial_t \\ -\nabla \end{pmatrix} u,$$

where we used (once again !) the Neumann boundary condition. As such, our two remaining terms compensate exactly if $\mu(Z \cdot N_1) = -1$: but as

$$Z - T = (Z \cdot N_1)N_1 = \partial_n + (Z \cdot N_2)N_2 \text{ implies } (Z \cdot N_1)(N_1 \cdot \partial_n) = 1,$$

we do get the desired result, namely $(\nu \cdot N_2)(Z \cdot N_1) = -1$.

As such, we have disposed with all the boundary terms and this achieves the proof of Proposition 3.1. \square

3.1.3. The L^6 estimate. — We are now in position to prove the classical non concentration effect:

Proposition 3.7. — *Assume that $x_0 \in \overline{\Omega}$, and u is a solution to (1.1) in the space $X_{<t_0}$, then*

$$(3.33) \quad \lim_{t \rightarrow t_0} \int_{D_t} u^6(t, x) dx = 0.$$

Proof. — We follow [14, 5, 10, 11] and simply have to take care of the boundary terms. We can assume that $x_0 \in \partial\Omega$ as otherwise these boundary terms disappear in the calculations below (which in this case are standard). Unlike in [12] we cannot use any convexity assumption to obtain that these terms have the right sign, but Proposition 3.1 will serve as a substitute. We may set $x_0 = 0$ and $t_0 = 0$ for convenience. Integrating over K_S^T the identity

$$0 = \operatorname{div}_{t,x}(tQ + u\partial_t u, -tP) + \frac{|u|^6}{3},$$

we get (see [12, (3.9)–(3.12)]),

$$\begin{aligned} 0 = \int_{D_T} (TQ + u\partial_t u)(T, x) dx - \int_{D_S} (SQ + u\partial_t u)(S, x) dx + \frac{1}{\sqrt{2}} \int_{M_S^T} (tQ + u\partial_t u + x \cdot P) d\rho \\ - \int_{((S,T) \times \partial\Omega) \cap K_S^T} \nu(x) \cdot (tP) d\sigma + \frac{1}{3} \int_{K_S^T} u^6 dx dt. \end{aligned}$$

Using Hölder's inequality and the conservation of energy, we get that the first term in the left is controlled by CTE_0 , whereas the last term is non negative. This yields

$$\begin{aligned} (3.34) \quad - \int_{D_S} (SQ + u\partial_t u)(S, x) dx + \frac{1}{\sqrt{2}} \int_{M_S^T} (tQ + u\partial_t u + x \cdot P) d\rho \\ \lesssim \int_{((S,0) \times \partial\Omega) \cap K_S^T} \nu(x) \cdot (tP) d\sigma + TE_0. \end{aligned}$$

By direct calculation (see [12, (3.11)]),

$$(3.35) \quad \frac{1}{\sqrt{2}} \int_{M_S^T} (tQ + u\partial_t u + x \cdot P) d\rho \\ = \frac{1}{\sqrt{2}} \int_{M_S^T} \frac{1}{t} |t\partial_t u + x \cdot \nabla_x u + u|^2 d\rho + \frac{1}{2} \int_{\partial D_S} u^2(S, x) d\sigma - \frac{1}{2} \int_{\partial D_T} u^2(T, x) d\sigma.$$

By the trace theorem and Hölder,

$$\int_{\partial D_T} u^2(T, x) d\sigma \lesssim T \|u(T)\|_{H^1(D_T)}^2 \lesssim TE_0.$$

On the other hand (see [12, (3.12)])

$$(3.36) \quad - \int_{D_S} (SQ + u\partial_t u)(S, x) dx \geq -\frac{1}{2} \int_{\partial D_S} u^2 d\sigma - S \int_{D_S} \frac{|u|^6(S, x)}{6} dx.$$

As a consequence, we obtain

$$(3.37) \quad (-S) \int_{D_S} \frac{|u|^6(S, x)}{6} dx + \frac{1}{\sqrt{2}} \int_{M_S^T} \frac{1}{t} |t\partial_t u + x \cdot \nabla_x u + u|^2 d\sigma(x, t) \\ \lesssim \int_{\partial\Omega_S^T \cap K_S^T} n(x) \cdot tP d\sigma dt + TE_0,$$

Taking (3.4) into account (and the Neumann boundary condition), we obtain on $\partial\Omega$

$$tn(x) \cdot P = \frac{1}{2} (n(x) \cdot x) \left(\frac{|\partial_t u|^2 - |\partial_x u|^2}{2} + \frac{|u|^6}{6} \right).$$

However, for $x \in \partial\Omega$, given that $x_0 = 0 \in \partial\Omega$, we have

$$\frac{x}{|x|} = T + \mathcal{O}(x), \quad n(x) = n(0) + \mathcal{O}(x)$$

where T is a unit vector tangent to $\partial\Omega$ at $x_0 = 0$. Consequently, as $n(0) \cdot T = 0$,

$$n(x) \cdot x = \mathcal{O}(|x|^2), \text{ for } x \in \partial\Omega$$

and the right hand side in (3.37) is bounded (using Proposition 3.1) by

$$(3.38) \quad |S|^2(E_0 + E_0^{\frac{2}{3}}) + TE_0.$$

Sending T to zero, the second term disappears, and after dividing by $-S$, we get

$$(3.39) \quad \int_{D_S} |u|^6 dx \lesssim |S|(E_0 + E_0^{\frac{2}{3}}) + \frac{1}{\sqrt{2}|S|} \int_{M_S^0} \frac{1}{|t|} |t\partial_t u + x \cdot \nabla_x u + u|^2 d\sigma(x, t);$$

finally, by Hölder's inequality and (3.6), we obtain

$$\frac{1}{\sqrt{2}|S|} \int_{M_S^0} \frac{1}{|t|} |t\partial_t u + x \cdot \nabla_x u + u|^2 d\sigma(x, t) \leq \sqrt{2} \int_{M_S^0} \frac{|x|}{|S|} \left| \frac{x}{|x|} \partial_t u - \nabla_x u \right|^2 d\sigma(x, t) \\ + \sqrt{2} \int_{M_S^0} \frac{|u|^2}{|S||t|} d\sigma(x, t) \\ \lesssim \text{Flux}(u, M_S^0) + \text{Flux}(u, M_S^0)^{1/3},$$

hence,

$$(3.40) \quad \int_{D_S} |u|^6(S, x) dx \lesssim |S|(E_0 + E_0^{\frac{2}{3}}) + \text{Flux}(u, M_S^0) + \text{Flux}(u, M_S^0)^{1/3}$$

which is exactly (3.33) thanks to (3.9). Remark that in the calculations above all integrals on K_S^0 and M_S^0 have to be understood as the limits as $T \rightarrow 0^-$ of the respective integrals on K_S^T and M_S^T (which exist according to (3.6), (3.8)). \square

3.2. Global existence. — Once one has obtained the non-concentration result, Proposition 3.7, the remaining part of the proof follows very closely [3] and we reproduce it only to be self-contained. We consider u , the unique forward maximal solution to the Cauchy problem (1.1) in the space $X_{<t_0}$. Assume that $t_0 < +\infty$ and consider a point $x_0 \in \bar{\Omega}$; our aim is to prove that u can be extended in a neighborhood of (x_0, t_0) , which will imply a contradiction. Up to a space time translation, we set $(x_0, t_0) = (0, 0)$.

3.2.1. Localizing space-time estimates. — For $t < t' \leq 0$, let us denote by

$$\|u\|_{(L^p; L^q)(K_t^{t'})} = \left(\int_{s=t}^{t'} \left(\int_{\{|x| < -s\} \cap \Omega} |u|^q(s, x) dx \right)^{\frac{p}{q}} ds \right)^{\frac{1}{p}}$$

the $L_t^p L_x^q$ norm on $K_t^{t'}$ (with the usual modification if p or q is infinite). Similarly one defines space-time norms on boundaries.

Our main result in this section reads

Proposition 3.8. — *For any $\varepsilon > 0$, there exists $t < 0$ such that*

$$(3.41) \quad \|u\|_{(L^5; L^{10})(K_t^0)} < \varepsilon.$$

Proof. — We start with an extension result from [3] which applies equally in our setting:

Lemma 3.9. — *For any $x_0 \in \bar{\Omega}$ there exists $r_0 > 0$ such that for any $0 < r < r_0$ and any $v \in H^1(\Omega) \cap L^p(\Omega)$, there exist a function $\tilde{v}_r \in H^1(\Omega)$ (independent of the choice of $1 \leq p \leq +\infty$), satisfying*

$$(3.42) \quad \begin{aligned} & (\tilde{v}_r - v)|_{|x-x_0| < r \cap \Omega} = 0, \\ & \int_{\Omega} |\nabla \tilde{v}|^2 \lesssim \int_{\Omega} |\nabla v|^2, \quad \|\tilde{v}_r\|_{L^p(\Omega)} \lesssim \|v\|_{L^p(\{|x-x_0| < r\})}. \end{aligned}$$

In other words, we can extend functions in $H^1 \cap L^p$ on the ball $\{|x - x_0| < r\}$ to functions in $H^1(\Omega) \cap L^p(\Omega)$ with uniform bounds with respect to (small) $r > 0$, for the H^1 and the L^p norms respectively.

Furthermore, for any $u \in L^\infty((-1, 0); H_N^1(\Omega)) \cap L_{loc}^1((-1, 0); L^p(\Omega))$, there exist a function $\tilde{u} \in L^\infty((-1, 0); H_N^1(\Omega)) \cap L_{loc}^1((-1, 0); L^p(\Omega))$, satisfying (uniformly with respect to t)

$$(3.43) \quad \begin{aligned} & (\tilde{u} - u)|_{\{|x-x_0| < -t\} \cap \Omega} = 0, \\ & \int_{\Omega} |\nabla \tilde{u}|^2(t, x) + |\partial_t \tilde{u}|^2(t, x) dx \lesssim \int_{\Omega} |\nabla u|^2(t, x) + |\partial_t u|^2(t, x) \\ & \|\tilde{u}(t, \cdot)\|_{L^p(\Omega)} \lesssim \|u(t, \cdot)\|_{L^p(\Omega \cap \{|x-x_0| < -t\})} \quad t\text{-a.s.} \end{aligned}$$

Proof. — See [3] where the boundary condition is easily modified to be Neumann rather than Dirichlet. \square

Let us come back to the proof of Proposition 3.8. Let \check{u} be the function given by the second part of Lemma 3.9. Then $(\check{u})^5$ is equal to u^5 on K_t^0 and

$$(3.44) \quad \begin{aligned} \|(\check{u})^5\|_{L^{\frac{5}{4}}((t,t');L^{\frac{30}{17}}(\Omega))} &\leq \|\check{u}\|_{L^5((t,t');L^{10}(\Omega))}^4 \|\check{u}\|_{L^\infty((t,t');L^6(\Omega))} \\ &\lesssim \|u\|_{(L^5;L^{10})(K_t^{t'})}^4 \|u\|_{(L^\infty;L^6)(K_t^0)}. \end{aligned}$$

On the other hand, $\nabla_x(\check{u})^5 = 5(\check{u})^4 \nabla_x \check{u}$ and

$$(3.45) \quad \begin{aligned} \|\nabla_x(\check{u})^5\|_{L^{\frac{5}{4}}((t,t');L^{\frac{10}{9}}(\Omega))} &\leq 5\|\check{u}\|_{L^5((t,t');L^{10}(\Omega))}^4 \|\nabla_x \check{u}\|_{L^\infty((t,0);L^2(\Omega))} \\ &\lesssim \|u\|_{(L^5;L^{10})(K_t^{t'})}^4 \|u\|_{L^\infty;H^1(\Omega)}. \end{aligned}$$

By (complex) interpolation, as in (3.2),

$$\|(\check{u})^5\|_{L^{\frac{5}{4}}((t,t');W^{\frac{7}{10},\frac{5}{4}}(\Omega))} \lesssim \|u\|_{(L^5;L^{10})(K_t^{t'})}^4 \|u\|_{L^\infty;H^1(\Omega)}^{\frac{7}{10}} \|u\|_{(L^\infty;L^6)(K_t^0)}^{\frac{3}{10}}.$$

Let w be the solution (which, by finite speed of propagation, coincides with u on K_t^0) of

$$(\partial_s^2 - \Delta)w = -(\check{u})^5, \quad \partial_n w|_{\partial\Omega} = 0, \quad (w - u)|_{s=t} = \partial_s(w - u)|_{s=t} = 0,$$

applying (2.3), and the Sobolev embedding $W^{\frac{3}{10},5}(\Omega) \hookrightarrow L^{10}(\Omega)$, we get

$$(3.46) \quad \begin{aligned} \|u\|_{(L^5;L^{10})(K_t^{t'})} &\lesssim (\|w\|_{(L^5((t,t');W^{\frac{3}{10},5}(\Omega))}) \\ &\lesssim E(u) + \|u\|_{(L^5;L^{10})(K_t^{t'})}^4 \|u\|_{L^\infty;H^1(\Omega)}^{\frac{7}{10}} \|u\|_{(L^\infty;L^6)(K_t^0)}^{\frac{3}{10}}. \end{aligned}$$

Finally, from Proposition 3.7, (3.46) and the continuity of the mapping $t' \in [t, 0) \rightarrow \|u\|_{(L^5;L^{10})(K_t^{t'})}$ (which takes value 0 for $t' = t$), there exists t (close to 0) such that

$$\forall t < t' < 0; \quad \|u\|_{(L^5;L^{10})(K_t^{t'})} \lesssim E(u)$$

and passing to the limit $t' \rightarrow 0$,

$$\|u\|_{(L^5;L^{10})(K_t^0)} \leq 2CE(u).$$

As a consequence, taking $t < 0$ even smaller if necessary, we obtain

$$(3.47) \quad \|u\|_{(L^5;L^{10})(K_t^0)} \leq \varepsilon.$$

\square

3.2.2. Global existence. — We are now ready to prove the global existence result. Let $t < t_0 = 0$ be close to 0 and let v be the solution to the linear equation

$$(\partial_s^2 - \Delta)v = 0, \quad \partial_n v|_{\partial\Omega} = 0, \quad (v - u)|_{s=t} = 0, \quad \partial_s(v - u)|_{s=t} = 0,$$

then the difference $w = u - v$ satisfies

$$(\partial_s^2 - \Delta)w = -u^5, \quad \partial_n w|_{\partial\Omega} = 0, \quad w|_{s=t} = 0, \quad \partial_s w|_{s=t} = 0.$$

Let \check{u} be the function given by Lemma 3.9 from u . We have

$$\|\check{u}\|_{L^5((t,0);L^{10}(\Omega))} \lesssim \varepsilon, \quad \|\check{u}\|_{L^\infty;H^1} \lesssim E(u).$$

Let \tilde{w} be the solution to

$$(\partial_s^2 - \Delta)\tilde{w} = -\tilde{u}^5, \quad \partial_n \tilde{w}|_{\partial\Omega} = 0, \quad \tilde{w}|_{s=t} = 0, \quad \partial_s \tilde{w}|_{s=t} = 0.$$

By finite speed of propagation, w and \tilde{w} coincide in K_t^0 . On the other hand, using (2.4) yields

$$(3.48) \quad \|\tilde{w}\|_{L^\infty((t,0);H^1)} + \|\partial_s \tilde{w}\|_{L^\infty((t,0);L^2(\Omega))} + \|\tilde{w}\|_{L^5((t,0);W^{\frac{3}{10},5}(\Omega))} \\ \lesssim \|\tilde{u}^5\|_{L^1((t,0);L^2(\Omega))} \lesssim \|\tilde{u}\|_{L^5((t,0);L^{10}(\Omega))}^5 \lesssim \varepsilon^5.$$

Finally, for any ball D , denote by

$$E(f(s, \cdot), D) = \int_{D \cap \Omega} (|\nabla_x f|^2 + |\partial_s f|^2 + \frac{|f|^6}{3})(s, x) dx;$$

since v is a solution to the linear equation,

$$(3.49) \quad E(v(s, \cdot), D(x_0 = 0, -s)) \rightarrow 0, \quad s \rightarrow 0^-$$

Recalling that $u = v + \tilde{w}$ inside K_t^0 , we obtain from (3.49) and (3.48) (and the Sobolev injection $H_N^1(\Omega \rightarrow L^6(\Omega))$) that there exists a small $s < 0$ such that

$$E(u(s, \cdot), D(x_0 = 0, -s)) < \varepsilon;$$

but, since $(u, \partial_s u)(s, \cdot) \in H_N^1(\Omega) \times L^2(\Omega)$, we have, by dominated convergence,

$$E(u(s, \cdot), D(x_0 = 0, -s)) = \int_{\Omega} 1_{\{|x-x_0| < -s\}}(x) (|\nabla u(s, x)|^2 + |\partial_s u(s, x)|^2 + \frac{|u|^6(s, x)}{3}) dx \\ \text{and } \lim_{\alpha \rightarrow 0} \int_{\Omega} 1_{\{|x-x_0| < \alpha-s\}}(x) (|\nabla u(s, x)|^2 + |\partial_s u(s, x)|^2 + \frac{|u|^6(s, x)}{3}) dx \\ = \lim_{\alpha \rightarrow 0} E(u(s, \cdot), D(x_0 = 0, -s + \alpha));$$

consequently, there exists $\alpha > 0$ such that

$$E(u(s, \cdot), D(x_0 = 0, -s + \alpha)) \leq 2\varepsilon.$$

Now, according to (3.7), the L^6 norm of u remains smaller than 2ε on $\{|x-x_0| < \alpha-s'\}$, $s \leq s' < 0$. As a consequence, the same proof as for Proposition 3.8 shows that the $L^5; L^{10}$ norm of the solution on the truncated cone

$$K = \{(x, s'); |x-x_0| < \alpha-s', s < s' < 0\}$$

is bounded. Since this is true for all $x_0 \in \overline{\Omega}$, a compactness argument shows that

$$\|u\|_{L^5((s,0);L^{10}(\Omega))} < +\infty$$

which, by Duhamel formula shows that

$$\lim_{s' \rightarrow 0^-} (u, \partial_s u)(s', \cdot)$$

exists in $(H_N^1(\Omega) \times L^2(\Omega))$ and consequently u can be extended for $s' > 0 = t_0$ small enough, using Corollary 2.2.

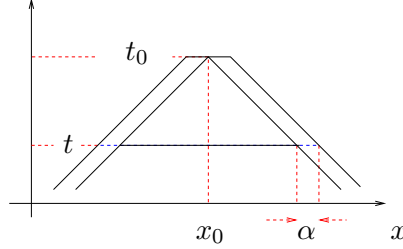


FIGURE 1. The truncated cone

Appendix

To prove Proposition 2.1, we observe that according to Theorems A and 2, the operator $\mathcal{T} = e^{\pm it\sqrt{-\Delta_N}}$ satisfies

$$(3.50) \quad \|\mathcal{T}u_0\|_{L^5((0,1)\times\Omega)} \lesssim \|u_0\|_{H_N^{\frac{7}{10}}(\Omega)}.$$

Applying the previous inequality to Δu_0 , and using the L^p elliptic regularity result

$$(3.51) \quad \begin{aligned} -\Delta u + u = f \in L^p(\Omega), \quad \partial_n u|_{\partial\Omega} = 0 &\Rightarrow u \in W^{2,p}(\Omega) \cap W_N^{1,p}(\Omega) \\ \text{and } \|u\|_{W^{2,p}(\Omega)} &\lesssim \|f\|_{L^p(\Omega)}, \quad 1 < p < +\infty \end{aligned}$$

we get

$$(3.52) \quad \|\mathcal{T}u_0\|_{L^5((0,1);W^{2,5}(\Omega)\cap W_N^{1,5}(\Omega))} \lesssim \|u_0\|_{H_N^{\frac{27}{10}}(\Omega)}$$

and consequently by (complex) interpolation between (3.50) and (3.52),

$$(3.53) \quad \|\mathcal{T}u_0\|_{L^5((0,1);W_N^{\frac{3}{10},5}(\Omega))} \lesssim \|u_0\|_{H_N^1(\Omega)};$$

finally, by Sobolev embedding

$$(3.54) \quad \|\mathcal{T}u_0\|_{L^5((0,1);L^{10}(\Omega))} \lesssim \|u_0\|_{H_N^1(\Omega)}.$$

To conclude, we simply observe that

$$u = \cos(t\sqrt{-\Delta_N})u_0 + \frac{\sin(t\sqrt{-\Delta_N})}{\sqrt{-\Delta_N}}u_1$$

and $1/\sqrt{-\Delta_N}$ is an isometry from $L^2(\Omega)$ to $H_N^1(\Omega)$, providing the result.

The usual TT^* argument and Christ-Kiselev Lemma [4] proves the following:

Proof. — We have

$$u(t, \cdot) = \cos(t\sqrt{-\Delta_N})u_0 + \frac{\sin(t\sqrt{-\Delta_N})}{\sqrt{-\Delta_N}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta_N})}{\sqrt{-\Delta_N}}f(s, \cdot)ds.$$

The contributions of (u_0, u_1) are easily dealt with, as previously. Let us focus on the contribution of

$$\int_0^t \frac{e^{i(t-s)\sqrt{-\Delta_N}}}{\sqrt{-\Delta_N}}f(s, \cdot)ds.$$

Denote by $\mathcal{T} = e^{it\sqrt{-\Delta_N}}$; interpolating between (3.50) and (3.52),

$$\|\mathcal{T}f\|_{L^5((0,1);W^{2-\frac{7}{10},5}(\Omega)\cap W_N^{1,5}(\Omega))} \lesssim \|f\|_{H_N^2(\Omega)}.$$

Let $u_0 \in L^2$. Then there exist $v_0 \in H_N^2(\Omega)$ such that

$$-\Delta v_0 = u_0, \quad \|u_0\|_{L^2} \sim \|v_0\|_{H^2};$$

as a consequence, from $\mathcal{T}u_0 = \Delta \mathcal{T}v_0$,

$$\begin{aligned} \|\mathcal{T}u_0\|_{L^5((0,1);W^{-\frac{7}{10},5}(\Omega))} &\lesssim \|\mathcal{T}v_0\|_{L^5((0,1);W^{2-\frac{7}{10},5}(\Omega)) \cap W_N^{1,5}(\Omega)} \\ &\lesssim \|v_0\|_{H_N^2(\Omega)} \sim \|u_0\|_{L^2(\Omega)}. \end{aligned}$$

By duality we deduce that the operator \mathcal{T}^* defined by

$$\mathcal{T}^*f = \int_0^1 e^{-is\sqrt{-\Delta}} f(s, \cdot) ds$$

is bounded from $L^{\frac{5}{4}}((0,1);W^{\frac{7}{10},\frac{5}{4}}(\Omega))$ to $L^2(\Omega)$ (observe that $W^{\frac{7}{10},\frac{5}{4}}(\Omega) = W_N^{\frac{7}{10},\frac{5}{4}}(\Omega)$); using (3.53) and boundedness of $\sqrt{-\Delta_N}^{-1}$ from L^2 to H_N^1 , we obtain

$$\|\mathcal{T}(\sqrt{-\Delta_N})^{-1}\mathcal{T}^*f\|_{L^5((0,1);W_N^{\frac{3}{10},5}(\Omega)) \cap L^\infty((0,1);H_N^1(\Omega))} \lesssim \|f\|_{L^{\frac{5}{4}}((0,1);W_N^{\frac{7}{10},\frac{5}{4}}(\Omega))},$$

and

$$\|\partial_t \mathcal{T}(\sqrt{-\Delta_N})^{-1}\mathcal{T}^*f\|_{L^\infty((0,1);L^2(\Omega))} \leq C\|f\|_{L^{\frac{5}{4}}((0,1);W_N^{\frac{7}{10},\frac{5}{4}}(\Omega))}.$$

But

$$\mathcal{T}(\sqrt{-\Delta_N})^{-1}\mathcal{T}^*f(s, \cdot) = \int_0^1 \frac{e^{i(t-s)\sqrt{-\Delta_N}}}{\sqrt{-\Delta_N}} f(s, \cdot) ds$$

and an application of Christ-Kiselev lemma [4] allows to transfer this property to the operator

$$f \mapsto \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta_N}}}{\sqrt{-\Delta_N}} f(s, \cdot) ds.$$

□

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